

SOME EXAMPLES OF NON-TIDY SPACES

TAKAHIRO MATSUSHITA

ABSTRACT. We construct a free \mathbb{Z}_2 -manifold X_n for a positive integer n such that $w_1(X_n)^n \neq 0$, but there is no \mathbb{Z}_2 -equivariant map from S^2 to X_n .

1. INTRODUCTION AND MAIN THEOREMS

First we fix terminologies and notations we use in this paper. The group acts on spaces from the right unless otherwise stated. Let Γ denote a group. In this paper, the Γ -action on a space X is said to be free if for any $x \in X$, there is a neighborhood U of x such that $U\gamma \cap U = \emptyset$ for any $\gamma \in \Gamma \setminus \{e_\Gamma\}$. For a Γ -space X , we denote the orbit space of X by \overline{X} . Let $x_0 \in X$. The image of $\pi_1(X, x_0)$ via the group homomorphism $\pi_1(X, x_0) \rightarrow \pi_1(\overline{X}, \overline{x}_0)$ induced by the quotient $X \rightarrow \overline{X}$ is often written by $\pi_1(X, x_0)$ also, for simplicity. The coefficient of the singular cohomology is considered as \mathbb{Z}_2 , the cyclic group with order 2.

We write S_a^n for the n -dimensional sphere with the antipodal action. For a free \mathbb{Z}_2 -space X , we put

$$\begin{aligned} \text{coind}(X) &= \sup\{n \geq 0 \mid \text{There is a } \mathbb{Z}_2\text{-map from } S_a^n \text{ to } X.\}, \\ \text{ind}(X) &= \inf\{n \geq 0 \mid \text{There is a } \mathbb{Z}_2\text{-map from } X \text{ to } S_a^n.\}, \\ h(X) &= \sup\{n \geq 0 \mid w_1(X)^n \neq 0\} \end{aligned}$$

and call the coindex, the index, and the Stiefel-Whitney height of X respectively¹, where $w_1(X) \in H^1(\overline{X})$ is the 1st Stiefel-Whitney class of the double cover $X \rightarrow \overline{X}$. It is obvious that

$$\text{coind}(X) \leq h(X) \leq \text{ind}(X)$$

for every free \mathbb{Z}_2 -space X . In [1], X is said to be *tidy* if $\text{ind}(X) = \text{coind}(X)$.

In this paper, we prove the following.

Theorem 1.1. *For a positive integer n , there is a free \mathbb{Z}_2 -space X_n such that $h(X) = n$ but $\text{coind}(X_n) = 1$.*

The space X_n is defined as follows. Let S_b^n denote the \mathbb{Z}_2 - \mathbb{Z}_2 -space where its base space is the n -dimensional sphere $S^n \subset \mathbb{R}^{n+1}$, and the left and the right \mathbb{Z}_2 -actions are defined by

$$\begin{aligned} \tau(x_0, \dots, x_n) &= (-x_0, x_1, \dots, x_n), \\ (x_0, \dots, x_n)\tau &= (-x_0, \dots, -x_n), \end{aligned}$$

where τ is the generator of \mathbb{Z}_2 . Then we define $X_1 = S_a^1$, and $X_{k+1} = X_k \times_{\mathbb{Z}_2} S_b^1$.

In [2], Schultz proved that $h(X \times_{\mathbb{Z}_2} S_b^n) \geq h(X) + n$ for any free \mathbb{Z}_2 -space X . We prove the equality holds, although this is not necessary to prove Theorem 1.1.

Theorem 1.2. *$h(X \times_{\mathbb{Z}_2} S_b^n) = h(X) + n$ for any free \mathbb{Z}_2 -space X .*

¹These terminologies are due to [1]. Many different terminologies are used, see [3] or [4].

As far as I know, there is no explicit example published whose difference between the Stiefel-Whitney height and the coindex is greater than 1. On the other hand, it is known that the difference between $\text{ind}(X)$ and $h(X)$ can be arbitrarily large. Indeed, the odd dimensional real projective space $\mathbb{R}P^{2n-1}$ is such example by the result of Stolz [3], see also [4].

2. PROOFS

First we prove Theorem 1.1. As is said in Section 1, Schultz proved that $h(X \times_{\mathbb{Z}_2} S_b^n) \geq h(X) + n$. So we have that $h(X_n) \geq n$. Since \overline{X}_n is an n -dimensional manifold (or by Theorem 1.2), we have $h(X_n) = n$. So what we must show is that there is no \mathbb{Z}_2 -equivariant map from S_a^2 to X_n . To prove this, we establish a criterion to show the non-existence of equivariant maps, using fundamental groups.

Let Γ be a discrete group, and X a path-connected free Γ -space. Let $x_0 \in X$ be a base point. Then by the covering space theory, we have an isomorphism

$$\Gamma \cong \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0).$$

Recall that this isomorphism is given as follows. For $\alpha \in \pi_1(\overline{X}, \overline{x}_0)$, let $\varphi \in \alpha$ and let $\tilde{\varphi}$ denote the lift of φ whose initial point is x_0 . Then the terminal point of $\tilde{\varphi}$ is in the fiber over \overline{x}_0 , so there is a unique $\Phi_X(\alpha) \in \Gamma$ such that $\varphi(1) = x_0 \Phi_X(\alpha)$. This $\Phi_X : \pi_1(\overline{X}, \overline{x}_0) \rightarrow \Gamma$ is a group homomorphism, which is surjective since X is path-connected, and its kernel is $\pi_1(X, x_0)$. Hence Φ_X induces the isomorphism

$$\overline{\Phi}_X : \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0) \longrightarrow \Gamma.$$

Let X and Y be connected free Γ -spaces and $f : X \rightarrow Y$ a Γ -equivariant map. Let $x_0 \in X$ and put $y_0 = f(x_0)$. Since the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ q_{X*} \downarrow & & \downarrow q_{Y*} \\ \pi_1(\overline{X}, \overline{x}_0) & \xrightarrow{\tilde{f}_*} & \pi_1(\overline{Y}, \overline{y}_0) \end{array}$$

is commutative, so we have a group homomorphism

$$\hat{f}_* : \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0) \rightarrow \pi_1(\overline{Y}, \overline{y}_0) / \pi_1(Y, y_0).$$

We write $\Psi(f) : \Gamma \rightarrow \Gamma$ for the group homomorphism which commutes the following diagram.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Psi(f)} & \Gamma \\ \overline{\Phi}_X \uparrow & & \uparrow \overline{\Phi}_Y \\ \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0) & \xrightarrow{\hat{f}_*} & \pi_1(\overline{Y}, \overline{y}_0) / \pi_1(Y, y_0). \end{array}$$

Then we have the following.

Proposition 2.1. *The group homomorphism $\Psi(f) : \Gamma \rightarrow \Gamma$ is the identity.*

Proof. Let $\alpha \in \pi_1(\overline{X}, \overline{x}_0) / \pi_1(X, x_0)$ and let φ be the loop of $(\overline{X}, \overline{x}_0)$ which represents α . Let $\tilde{\varphi}$ denote the lift of φ whose initial point is x_0 . Then $\tilde{\varphi}(1) = x_0 \overline{\Phi}_X(\alpha)$. Then we have $y_0 \overline{\Phi}_X(\alpha) = f \circ \tilde{\varphi}(1) = y_0 \overline{\Phi}_Y(\hat{f}_* \alpha)$. Hence $\overline{\Phi}_X = \overline{\Phi}_Y \circ \hat{f}_*$. \square

The situation we used here is the case $\Gamma = \mathbb{Z}_2$. Let X be a path-connected free \mathbb{Z}_2 -space and $x_0 \in X$, we say $\alpha \in \pi_1(\overline{X}, \overline{x}_0)$ is said to be even if $\alpha \in \pi_1(X, x_0)$, and is said to be odd if α is not even. Then Proposition 2.1 asserts that for a \mathbb{Z}_2 -equivariant map $f : X \rightarrow Y$, the group homomorphism $\overline{f}_* : \pi_1(\overline{X}, \overline{x}_0) \rightarrow \pi_1(\overline{Y}, \overline{y}_0)$ preserves the parity of $\pi_1(\overline{X}, \overline{x}_0)$.

Let us start to the proof of Theorem 1.1.

Lemma 2.2. *The group $\pi_1(\overline{X}_n)$ has no non-trivial torsion elements.*

Proof. By the definition of X_n , \overline{X}_n is the orbit space of a free and isometrical $\pi_1(\overline{X}_n)$ -action on \mathbb{R}^n . Remark that for an affine map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $a_1, \dots, a_m \in \mathbb{R}$ with $\sum_{i=1}^m a_i = 1$, and $x_1, \dots, x_m \in \mathbb{R}^n$, we have

$$A\left(\sum_{i=1}^m a_i x_i\right) = \sum_{i=1}^m a_i (Ax_i).$$

Let $\alpha \in \pi_1(\overline{X}_n)$ be a non-trivial torsion element and its order is denoted by k . Let $x \in \mathbb{R}^n$. Then the point

$$y = \frac{1}{k} \sum_{i=1}^k x \alpha^i \in \mathbb{R}^n$$

is fixed by α . This is contradiction. \square

Hence to prove Theorem 1.1, it is sufficient to prove the following.

Theorem 2.3. *Let X be a path-connected free \mathbb{Z}_2 -space. Then there is a \mathbb{Z}_2 -equivariant map from S_a^2 to X if and only if there is $\alpha \in \pi_1(\overline{X})$ such that $\alpha^2 = 1$ and $\alpha \notin \pi_1(X)$.*

Proof. Suppose there is a \mathbb{Z}_2 -map $f : S_a^2 \rightarrow X$. Let β denote the generator of $\pi_1(\overline{S}_a^2) \cong \mathbb{Z}_2$. Since β is odd, $\overline{f}_*(\beta)$ is odd. Since $\overline{f}_*(\beta) \cdot \overline{f}_*(\beta) = \overline{f}_*(\beta \cdot \beta) = 1$, we have completed the “only if” part.

On the other hand, suppose $\alpha \in \pi_1(\overline{X}, \overline{x}_0)$ which is odd and $\alpha^2 = 1$. Let $\varphi \in \alpha$, and let $\tilde{\varphi}$ denote the lift of φ whose initial point is x_0 . Then $\psi = (\tilde{\varphi}\tau) \cdot \tilde{\varphi}$ is a loop of (X, x_0) , and is null-homotopic since $q_{X*}[\psi] = \alpha^2 = 1$ and q_{X*} is injective, where $q_X : X \rightarrow \overline{X}$ is the quotient map. We can regard ψ is a \mathbb{Z}_2 -map from S_a^1 to X , and hence we can extend it to a \mathbb{Z}_2 -map $S_a^2 \rightarrow X$. This completes the proof. \square

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. It is sufficient to prove that $h(X \times_{\mathbb{Z}_2} S_b^{n+1}) = h(X \times_{\mathbb{Z}_2} S_b^n) + 1$ for $n \geq 0$. The part “ \geq ” is proved by Schultz in [2], so we give the proof of the part “ \leq ”. Put

$$A = \{(x_0, \dots, x_{n+1}) \in S_b^n \mid |x_{n+1}| \leq \frac{1}{2}\},$$

$$B = \{(x_0, \dots, x_{n+1}) \in S_b^n \mid |x_{n+1}| \geq \frac{1}{2}\}.$$

These are \mathbb{Z}_2 - \mathbb{Z}_2 -closed subset of S_b^{n+1} . Put $X' = X \times_{\mathbb{Z}_2} S_b^{n+1}$, $A' = X \times_{\mathbb{Z}_2} A$, and $B' = X \times_{\mathbb{Z}_2} B$. Then the followings hold.

- (1) $A' \simeq_{\mathbb{Z}_2} X \times_{\mathbb{Z}_2} S_b^n$.
- (2) $B' \simeq_{\mathbb{Z}_2} \overline{X} \sqcup \overline{X}$ with the involution exchanging each \overline{X} . Hence $w_1(B') = 0$.
- (3) $X' = A' \cup B'$.

Suppose $w_1(X \times_{\mathbb{Z}_2} S_b^n)^k = 0$. Then $w_1(A')^k = 0$. Then there is $\alpha \in H^k(X', A')$ which maps to $w_1(X')^k$ via $H^k(X', A') \rightarrow H^k(X')$. Similarly, there is $\beta \in H^1(X', B')$ which maps to $w_1(X')$ via $H^1(X', B') \rightarrow H^1(X')$. Then $\alpha \cup \beta \in H^{k+1}(X', A' \cup B') = 0$, and which maps to $w_1(X')^{k+1}$. Hence we have $w_1(X')^{k+1} = 0$. This completes the proof. \square

Acknowledgement. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

REFERENCES

- [1] J. Matoušek, *Using the Borsuk-Ulam theorem*, Universitext, Springer-Verlag Berlin Heidelberg (2003), corrected 2nd printing 2008.
- [2] C. Schultz, *Graph colorings, spaces of edges and spaces of circuits*, Adv. Math, **221**(6):1733-1756 (2009)
- [3] S. Stolz, *The level of real projective spaces*, Comment. Math. Helv., **64**(4):661-674 (1989)
- [4] R. Tanaka, *On the index and co-index of sphere bundles*, Kyushu J. Math. (**57**) 371-382, (2003)

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO, 153-8914 JAPAN

E-mail address: `tmatsu@ms.u-tokyo.ac.jp`